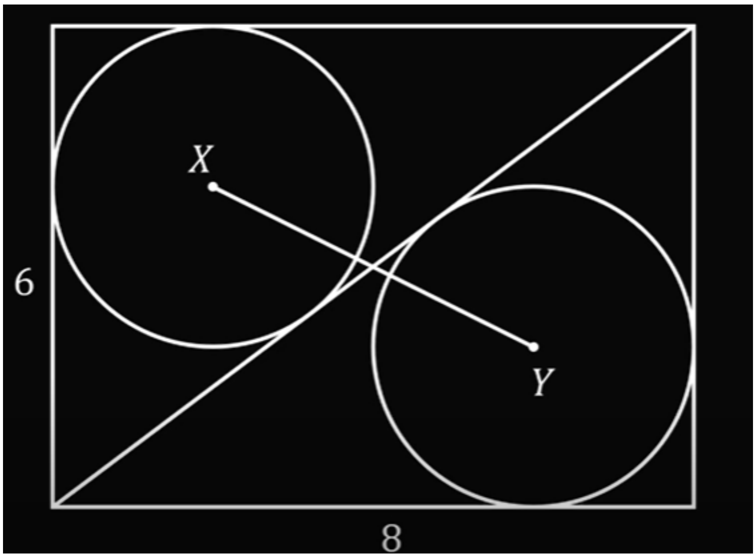
October Math Gems

Problem of the week 19

§1 problems

Problem 1.1. What is the length of $|XY|$



Problem 1.2. Suppose that α, β, and γ are the roots of the equation $x^3+3x^2-24x+1=0$. Find the value $\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}$

Problem 1.3. In the equation ax^2+5x+2, solve for a so that the equation has exactly one solution

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Problem 1.4. Given that $\frac{1}{\sqrt[3]{25}+\sqrt[3]{5}+1}=A\sqrt[3]{25}+B\sqrt[3]{5}+C$. So, what is the value of $A+B+C$ (A,B, and C are rational numbers).

Problem 1.5. The leg of a right-angled triangle is equal to $\frac{1}{5}$ the sum of the other sides, and the triangle's perimeter is 1. What is the area of the triangle?

Problem 1.6. There are 90 equally spaced dots marked on a circle. Shannon chooses an integer, n. Beginning at a randomly chosen dot, Shannon goes around the circle clockwise and colours in every nth dot. He continues going around and around the circle colouring in every nth dot, counting each dot whether it is coloured in or not, until he has coloured in every dot. Which of the following could have been Shannon's integer ? $(3,5,6,7)$

Problem 1.7. If $x^2=2023+y$, $y^2=2023+x$, where $x\neq y$ and x and y are both real numbers. Find the value of xy

Problem 1.8. Solve for real values of x

$$2\sqrt[3]{2x+1} = x^3 - 1$$

Problem 1.9. Simplify and prove your answer for real values of x

$$x = \sqrt[3]{8+3\sqrt{21}} + \sqrt[3]{8-3\sqrt{21}}$$

Problem 1.10. If in triangle ABC we have

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0,$$

show that $\sin^3 A + \sin^3 B + \sin^3 C = 3 \sin A \sin B \sin C$.

Problem 1.11. if ABC is a triangle and $\frac{AB}{cosC} = \frac{BC}{cosA} = \frac{AC}{cosB}$ then prove that ABC is equilateral.

Problem 1.12. if $\frac{4\sqrt{x^2+1}}{x} = \frac{5\sqrt{y^2+1}}{y} = \frac{6\sqrt{z^2+1}}{z}$ and $x + y + z = xyz$. find x , y and z.

Problem 1.13. Solve in real numbers the following equation: $(\sqrt{6} - \sqrt{5})^x + (\sqrt{3} - \sqrt{2})^x + (\sqrt{3} + \sqrt{2})^x + (\sqrt{6} + \sqrt{5})^x = 32$.

Problem 1.14. Solve in $\mathbb{R}$ the equation

$$\frac{1}{2^x + x + 1} + \frac{1}{3^x - 4x - 3} = \frac{1}{2^x - 4x - 2} + \frac{1}{x + 3^x}$$

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Problem 1.15. Find all real numbers a, b, c, d such that

$$\begin{cases} a + b + c + d = 20, \\ ab + ac + ad + bc + bd + cd = 150. \end{cases}$$

Problem 1.16. Let ABC be a right-angled triangle with $\angle ABC = 90°$, and let D be on AB such that $AD = 2DB$. What is the maximum possible value of $\angle ACD$?

Problem 1.17. In a triangle ABC, $\angle A = 2 \cdot \angle B$. Prove that $a^2 = b(b + c)$.

Problem 1.18.

$$\textit{Prove that}:$$

$$\sin^2 \alpha - \sin^2 \beta = \sin(\alpha + \beta) \sin(\alpha - \beta)$$

Problem 1.19. Determine all pairs (a, b) of real numbers for which there exists a unique symmetric 2×2 matrix M with real entries satisfying trace$(M) = a$ and det$(M) = b$.

Problem 1.20. Determine all pairs (a, b) of non-negative integers such that

$$\frac{a+b}{2} - \sqrt{ab} = 1.$$